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A Letter to the Student

The English philosopher and scientist Roger Bacon once wrote: "Mathematics is the gate and key of the sciences. . . . Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world."

In turn, algebra is the gate and key of mathematics. For this reason, colleges and universities require mastery of algebra in preparation for studying not only the sciences, but also such subjects as engineering, medicine, architecture, philosophy, psychology, and law.

Although many problems that can be solved by algebra can also be worked out by common sense, their translation into algebraic form generally makes them easier to deal with. Because of this, algebra has become the language of science. The goal of this course is to learn how to use this language.

Success in algebra depends on a combination of talent and effort. A few people are so gifted in mathematics that they can succeed with very little effort. For most people, however, diligent practice is the key to success. Like developing ability in a sport, becoming good at algebra takes practice. It is my hope that this book will help you both to enjoy the subject and to be successful in your studies.

Harold R. Jacobs

INTRODUCTION
A Number Trick

Think of a number from one to ten. Add seven to it. Multiply the result by two. Subtract four. Divide by two. Subtract the number that you first thought of. Is your answer five?

Number tricks such as this have long been popular. That the final result can be known by someone who doesn't know which number was originally chosen is surprising.

How does the trick work? If we make a table (like the one at the top of the next page) showing what happens when it is done with each number from one to ten, some patterns appear.

Would these patterns continue if the table were extended to include other numbers? If we began by thinking of eleven, would the answer at the end still be five? What if we began with one hundred? Would we get five at the end if we began with zero? Do you think it is correct to assume that the trick will work for *any* number you might think of?

Even though you may feel that the answer to every one of these questions is yes, *how* the trick works is still not clear. Merely doing arithmetic with a series of different numbers cannot reveal the secret of why they all lead to the same result.

The number thought of:	1	2	3	4	5	6	7	8	9	10
Add seven:	8	9	10	11	12	13	14	15	16	17
Multiply by two:	16	18	20	22	24	26	28	30	32	34
Subtract four:	12	14	16	18	20	22	24	26	28	30
Divide by two:	6	7	8	9	10	11	12	13	14	15
Subtract the number first thought of:	5	5	5	5	5	5	5	5	5	5

There is a simple way, however, to discover the secret. Instead of writing down a specific number at the start, we will use a symbol to represent whatever number might be chosen. We will begin with a box.

□

Throughout the trick this box will represent the number originally chosen.

The next step in the trick is to add seven. We will represent numbers we know with sets of circles, and so seven will look like this:

○ ○ ○ ○ ○ ○ ○

To show the result of adding seven to the number, we draw seven circles beside the box.

□ ○ ○ ○ ○ ○ ○ ○

If we illustrate the entire trick in this way, it looks like this:

The number thought of:	□
Add seven:	□ ○ ○ ○ ○ ○ ○ ○
Multiply by two:	□ ○ ○ ○ ○ ○ ○ ○ □ ○ ○ ○ ○ ○ ○ ○
Subtract four:	□ ○ ○ ○ ○ ○ □ ○ ○ ○ ○ ○
Divide by two:	□ ○ ○ ○ ○ ○
Subtract the number first thought of:	○ ○ ○ ○ ○

The pictures make it easy to see why, no matter what number we start with, the answer at the end of the trick is always five. The box representing the original number disappears in the last step, leaving five circles.

Doing arithmetic with symbols rather than specific numbers is the basis of algebra. The explanation with the boxes and circles of what is happening throughout the number trick is an example of this. One of our goals in learning algebra will be to learn how to set up and solve problems using symbols such as these.

Exercises

1. Here are directions for another number trick and part of a table to show what happens when the trick is done with each number from one to five.

Think of a number:	1	2	3	4	5
Double it:	2	4	[redacted]	[redacted]	[redacted]
Add six:	8	[redacted]	[redacted]	[redacted]	[redacted]
Divide by two:	4	[redacted]	[redacted]	[redacted]	[redacted]
Subtract the number that you first thought of:	3	[redacted]	[redacted]	[redacted]	[redacted]

a) Copy and complete the table.
b) Does your table prove that the trick will work for *any* number?
c) Show how the trick works by illustrating the steps with boxes and circles. The first two steps are shown below.

Think of a number: □
Double it: □□

d) Do your drawings prove that the trick will work for *any* number?

2. The pictures below illustrate the steps of another number trick. Tell what is happening in each step in words.

Step 1. □
Step 2. □□□□
Step 3. □□□□ ○○○○○○○○
Step 4. □ ○○
Step 5. □ ○○○○○
Step 6. ○○○○○

3. In the next number trick, we will study the effect of changing some of the directions.

Step 1.	Think of a number.
Step 2.	Add four.
Step 3.	Multiply by two.
Step 4.	Subtract four.
Step 5.	Divide by two.
Step 6.	Subtract the number that you first thought of.

a) What is the result at the end of this trick?

b) Suppose that the second step were changed as shown below.

Step 1.	Think of a number.
Step 2.	Add six.
Step 3.	Multiply by two.
Step 4.	Subtract four.
Step 5.	Divide by two.
Step 6.	Subtract the number that you first thought of.

The trick will still work, even though the result at the end is changed. How is it changed?

c) Suppose instead that the fourth step were changed as shown below.

Step 1.	Think of a number.
Step 2.	Add four.
Step 3.	Multiply by two.
Step 4.	Subtract six.
Step 5.	Divide by two.
Step 6.	Subtract the number that you first thought of.

What effect does this have on the trick?

d) Suppose instead that the third step were changed as shown below.

Step 1.	Think of a number.
Step 2.	Add four.
Step 3.	Multiply by four.
Step 4.	Subtract four.
Step 5.	Divide by two.
Step 6.	Subtract the number that you first thought of.

What effect does this have on the trick?

4. Here is the beginning of a number trick. Can you make up more steps so that it will give the same answer for any number a person might choose?

   Think of a number.
   Triple it.
   Add twelve.

"Can you do Addition?" the White Queen asked. "What's one and one and one and one and one and one and one and one and one and one?"
"I don't know," said Alice. "I lost count."

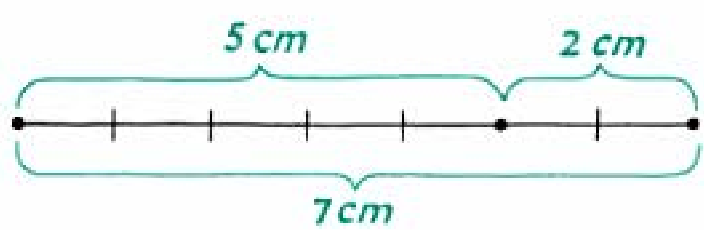
LEWIS CARROLL, *Through The Looking Glass*

LESSON 1
Addition

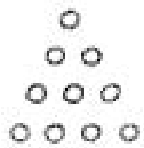
Soon after a child is able to count, he learns how to add. The two operations are closely connected, as anyone who has ever added by counting on his fingers knows. Consider the problem of adding the numbers represented by these two sets of circles:

At first a child finds the answer by counting all of the circles. Then he learns the fact that $5 + 2 = 7$.

Another way to picture addition is by lengths along a line. This figure also illustrates the fact that $5 + 2 = 7$.



The result of adding two or more numbers, called their **sum,** does not depend on either the order of the numbers or the order in which they are added. To find



the number of circles in the pattern above, for example, we could add the numbers of circles in the four rows from top to bottom:

$$1 + 2 + 3 + 4$$

or from bottom to top:

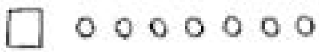
$$4 + 3 + 2 + 1$$

Either way, we get the same number: 10.

In algebra, it is often necessary to indicate the sum of two or more numbers without actually being able to add them. For example, in illustrating the number trick that appears in the introduction to this book, we used a box to represent the original number and a set of circles to represent the number seven:

Original number   Seven

To represent their sum, we drew the seven circles beside the box:



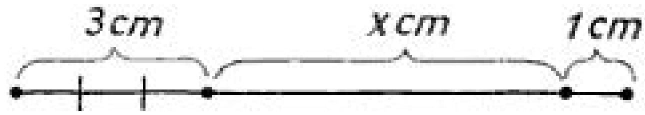
The sum of the original
number and seven

Instead of bothering to draw pictures like this, it is easier to represent the original number with a letter, such as x, and simply write

$$x + 7$$

The expression $x + 7$ means "the sum of x and 7." If we replace x with 1, $x + 7 = 1 + 7 = 8$. If we replace x with 2, $x + 7 = 2 + 7 = 9$, and so forth. Because x can be replaced by various numbers, it is called a **variable**.

If we know both numbers being added, such as 4 and 5, we can write their sum as a number, 9. If we know only one number or neither one, the best that we can do is to write an expression such as $x + 2$ or $x + y$. The length of the line segment below, for example, is the sum of the lengths of the three marked segments.



To indicate this sum, we can write $3 + x + 1$ or, more briefly, $x + 4$. Without knowing the length labeled x, we cannot simplify this answer any further.

Exercises

TEACHER: Haven't you finished adding up those numbers yet?
STUDENT: Oh, yes. I've added them up ten times already.
TEACHER: Excellent! I like a student who is thorough.
STUDENT: Thank you. Here are the ten answers.*

Set I

Find each of the following sums.

1. $1000 + 700 + 70 + 6$
2. $999 + 99 + 9$
3. $1 + 0.9 + 0.08 + 0.004$
4. $20 + 0.2 + 0.002$
5. $1 + 12 + 123 + 1234$
6. $1111 + 222 + 33 + 4$
7. $1 + 1.2 + 1.23 + 1.234$
8. $1.111 + 2.22 + 3.3 + 4$
9. $0.7 + 0.70 + 0.700 + 0.7000$
10. $0.5 + 0.55 + 0.555$

Set II

11. Write a number or expression for each of the following.

a) The sum of 10 and 7.
b) The sum of x and 7.
c) The sum of 10 and y.
d) The sum of x and y.
e) Four added to 8.
f) Four added to z.
g) The sum of 2, 5, and 1.
h) The sum of x, 5, and 1.
i) The sum of 2, y, and 1.
j) The sum of x, y, and 1.

* Alan Wayne, in *Mathematical Circles Revisited* by Howard W. Eves. © Copyright Prindle, Weber & Schmidt, Inc. 1971.

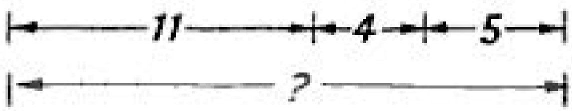
12. In the figures below, the box represents any number and the sets of circles represent specific numbers.

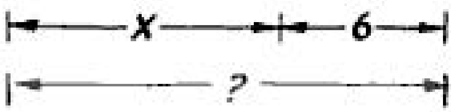
Figure 1

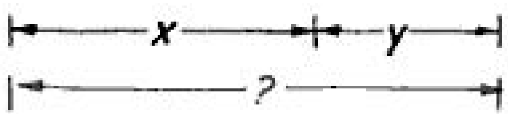
Figure 2

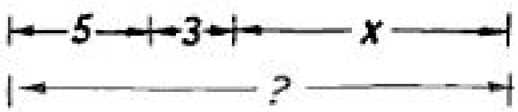
a) What addition problem is illustrated by Figure 1?
b) What is the answer to the problem?
c) Write an algebraic expression to represent the addition problem illustrated by Figure 2.
d) What is the answer to the problem if the box represents 2?
e) What is the answer to the problem if the box represents 4?

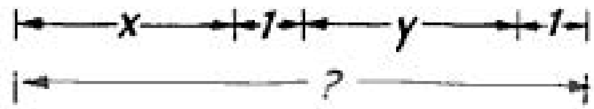
13. What is the length marked with a question mark in each of these figures?

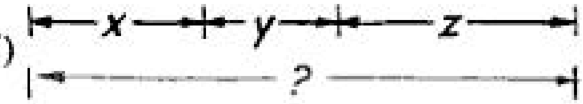
a)


b)


c)


d)


e)


f)


14. The figure below can be used to show that $3 + 7$ and $7 + 3$ are the same number, depending on whether the figure is read from left to right or from right to left.

Draw boxes and circles to show that
a) $x + 6$ and $6 + x$ mean the same thing.
b) $2 + x + 5$ and $x + 7$ mean the same thing.
c) $x + 4 + x$ and $4 + x + x$ mean the same thing.

15. The expression $x + y + 2$ represents the sum of x, y, and 2. If x is 1, it can be written as $1 + y + 2$ or $y + 3$. How can $x + y + 2$ be written if
a) x is 8?
b) x is 9?
c) y is 3?
d) y is 0?
e) x is 6 and y is 2?

16. Mr. Benny is 39 years old.
a) How old will he be in 5 years?
b) How old will he be in x years?
c) How old will he be 6 years after that?
Mrs. Benny is x years old.
d) How old will she be in 5 years?
e) How old will she be in y years?
f) How old will she be z years after that?

Set III

17. Write a number or expression for each of the following.
    a) The sum of 3 and 11.
    b) The sum of 3 and x.
    c) The sum of y and 11.
    d) The sum of y and x.
    e) Seven increased by 2.
    f) Seven increased by x.
    g) The sum of 9, 1, and 4.
    h) The sum of x, 1, and 4.
    i) The sum of 9, y, and 4.
    j) The sum of x, y, and 4.

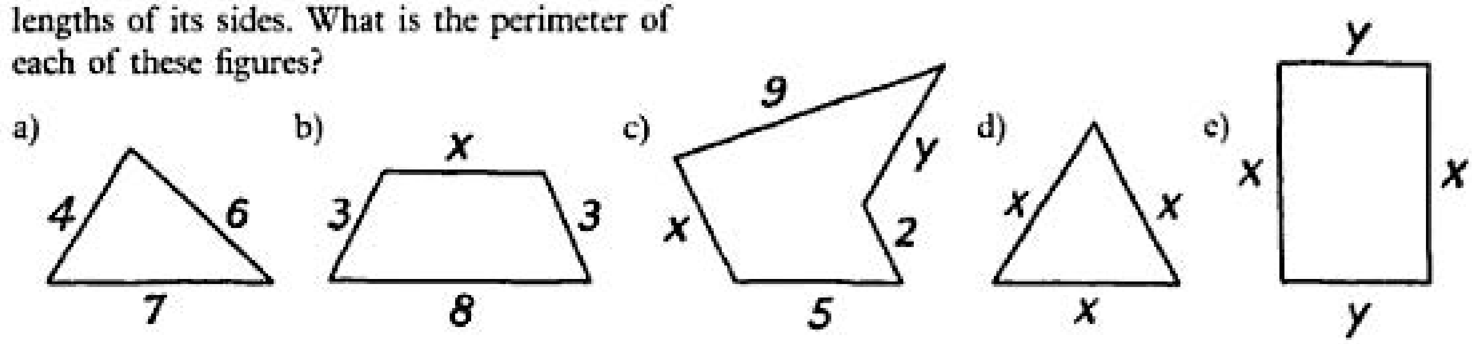
18. In the figures below, the box represents any number and the sets of circles represent specific numbers.

Figure 1

Figure 2

    a) What addition problem is illustrated by Figure 1?
    b) What is the answer to the problem?
    c) Write an algebraic expression to represent the addition problem illustrated by Figure 2.
    d) What is the answer to the problem if the box represents 1?
    e) What is the answer to the problem if the box represents 5?

19. The *perimeter* of a figure is the sum of the lengths of its sides. What is the perimeter of each of these figures?



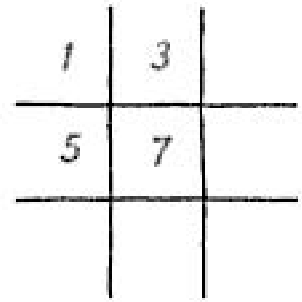
20. The figure below can be used to show that $4 + 5$ and $5 + 4$ are the same number, depending on whether the figure is read from left to right or from right to left.

Draw boxes and circles to show that
    a) $2 + x$ and $x + 2$ mean the same thing.
    b) $8 + x + 1$ and $x + 9$ mean the same thing.
    c) $x + x + 3$ and $x + 3 + x$ mean the same thing.

21. The expression $x + 1 + y$ represents the sum of x, 1, and y. If x is 4, it can be written as $4 + 1 + y$ or $5 + y$. How can $x + 1 + y$ be written if
    a) x is 2?
    b) x is 0?
    c) y is 6?
    d) y is 9?
    e) x is 3 and y is 7?

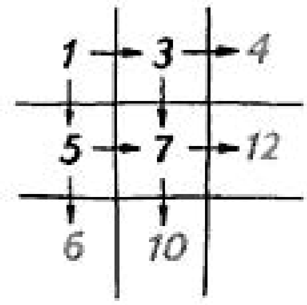
22. Each week, Dashing Dan jogs one mile farther than he did the week before.
    a) If he jogs 18 miles this week, how far will he jog next week?
    b) If he jogs x miles this week, how far will he jog next week?
    c) If he jogged y miles three weeks ago, how far will he jog this week?
    d) If he jogged y miles z weeks ago, how far will he jog this week?

Set IV A Number Puzzle



Numbers have been written in four spaces in this tic-tac-toe design. If we add across the rows and down the columns, we get the sums shown in the second figure. If we now add across the bottom row and down the last column, the answers are the same number:

$$6 + 10 = 16 \quad \text{and} \quad 4 + 12 = 16$$



Is this just a coincidence or would it happen if we started with *any* set of four numbers?

Draw a tic-tac-toe design and, in the same spaces as those in the example above, write four numbers of your own choosing. Add the rows and columns and see what happens. Can you explain why?